Effective Clipart Image Vectorization by Directly Optimizing Bezigons

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APPENDIX A

In this appendix we will introduce the complete definition of the rasterization function $R_{MS}(W;x,y)$, where W are the parameter set of a bezigon, and give a proof to illustrate the continuity and differentiability of this function with respect to the geometrical parameters.

A.1 Basic Definitions

Before describing the rasterization function, we introduce some basic definitions that will be needed throughout this section.

Based on [1], $R_{MS}(W;x,y)$ uses a hierarchical Haar wavelet representation to analytically calculate an anti-aliased raster image of a bezigon. Haar wavelets, as is well known, are represented by its mother wavelet function

$$\psi(t) = \begin{cases} 1, & t \in [0, \frac{1}{2}], \\ -1, & t \in [\frac{1}{2}, 1), \\ 0, & \text{otherwise.} \end{cases} \tag{A1}$$

and its scaling function

$$\phi(t) = \begin{cases} 1, & t \in [0, 1), \\ 0, & \text{otherwise.} \end{cases} \tag{A2}$$

Based on the above two functions, the 1D Haar basis with a scaling parameter $s \in \mathbb{Z}$ and a translating parameter $l \in \mathbb{Z}$ could be formally defined as

$$\psi_{s,k}(t) = \psi(2^s t - l), \quad t \in \mathbb{R}, \tag{A3}$$
$$\phi_{s,k}(t) = \phi(2^s t - l), \quad t \in \mathbb{R}. \tag{A4}$$

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Now let $k = (k_x, k_y) \in \mathbb{Z}^2$, the 2D Haar basis defined as following will be used later:

$$\psi_{s,k}^{(0,0)}(x,y) = 2^s \phi_{s,k_x}(x)\phi_{s,k_y}(y), \quad (x,y) \in \mathbb{R}^2, \tag{A5}$$
$$\psi_{s,k}^{(0,1)}(x,y) = 2^s \phi_{s,k_x}(x)\psi_{s,k_y}(y), \quad (x,y) \in \mathbb{R}^2, \tag{A6}$$
$$\psi_{s,k}^{(1,0)}(x,y) = 2^s \psi_{s,k_x}(x)\phi_{s,k_y}(y), \quad (x,y) \in \mathbb{R}^2, \tag{A7}$$
$$\psi_{s,k}^{(1,1)}(x,y) = 2^s \psi_{s,k_x}(x)\psi_{s,k_y}(y), \quad (x,y) \in \mathbb{R}^2. \tag{A8}$$

A.2 Rasterization Function $R_{MS}(W;x,y)$ And Its Continuity

According to [1], the value of pixel (x,y) in the raster image of a given 2D bezigon, indicated by the parameters $W = (B, C)$, takes the form

$$\begin{aligned} & R_{MS}(W;x,y) = \\ & c(C;x,y) \sum_{j=1}^{N} \left\{ \begin{array}{c} \sum_{k\in K} c_{0,k}^{(0,0)}(B;j)\psi_{0,k}^{(0,0)}(x,y) \\ + \sum_{s=0}^{d} \sum_{k\in K} \begin{bmatrix} c_{s,k}^{(0,1)}(B;j)\psi_{s,k}^{(0,1)}(x,y) \\ +c_{s,k}^{(1,0)}(B;j)\psi_{s,k}^{(1,0)}(x,y) \\ +c_{s,k}^{(1,1)}(B;j)\psi_{s,k}^{(1,1)}(x,y) \end{bmatrix} \end{array} \right\}, \\ & B \in \mathbb{R}^{6N}, C \in \mathbb{R}^3, (x,y) \in \Lambda, \\ & d \text{ is a given integer}, K \text{ is a finite set of } \mathbb{Z}^2. \end{aligned} \tag{A9}$$

Here, $c_{s,k}^{(\cdot)}(B;j)$ correspond to the wavelet coefficients contributed by the j-th Bézier curve segment:

$$\begin{aligned} c_{s,k}^{(0,0)}(B;j) &= \int_0^1 2^s \tilde{\phi}_{s,k_x}(X_j(B_x;t)) \\ &\quad \phi_{s,k_y}(Y_j(B_y;t))Y_j'(B_y;t)dt, \\ c_{s,k}^{(0,1)}(B;j) &= \int_0^1 -2^s \tilde{\psi}_{s,k_x}(Y_j(B_y;t)) \\ &\quad \phi_{s,k_y}(X_j(B_x;t))X_j'(B_x;t)dt, \\ c_{s,k}^{(1,0)}(B;j) &= \int_0^1 2^s \tilde{\psi}_{s,k_x}(X_j(B_x;t)) \\ &\quad \phi_{s,k_y}(Y_j(B_y;t))Y_j'(B_y;t)dt, \\ c_{s,k}^{(1,1)}(B;j) &= \int_0^1 2^s \tilde{\psi}_{s,k_x}(X_j(B_x;t)) \\ &\quad \psi_{s,k_y}(Y_j(B_y;t))Y_j'(B_y;t)dt. \end{aligned} \tag{A10}$$

The notations B_x, B_y and $X_j, Y_j (j = 1, 2, \ldots, N)$ are the same as Equation 2 and 3 in Section 3. Note that given the bezigon parameters B, both X_j and Y_j are functions of one variable t, while both X_j' and Y_j' are first-order derivatives with respect to t. For all $s \in \mathbb{Z}$ and $l \in \mathbb{Z}$,

$$\tilde{\phi}_{s,l}(t) = \int_0^t \phi_{s,l}(u)du, \qquad t \in \mathbb{R}, \tag{A11}$$

$$\tilde{\psi}_{s,l}(t) = \int_0^t \psi_{s,l}(u)du, \qquad t \in \mathbb{R}. \tag{A12}$$

It is obvious that both $\tilde{\phi}_{s,l}(t)$ and $\tilde{\psi}_{s,l}(t)$ are continuous with respect to the variable t respectively. Also, if $t = h(B)$ is a continuous function of any parameters of B, both $\tilde{\phi}_{s,l}(t)$ and $\tilde{\psi}_{s,l}(t)$ are too. From Equation 2, it is easy to see that both $X_j(B_x; t)$ and $Y_j(B_y; t)$ are continuous with respect to any parameters of B_x and B_y. Therefore, $c_{s,k}^{(\cdot)}(B; j)$ are also continuous with respect to B. Thus the continuity of $R_{MS}(W; x, y)$ with respect to geometrical parameters B is totally determined by above discussion and its formula A9. Such property is also reflected in Figure 2, where the data energy function using $R_{MS}(W; x, y)$ is continuous with respect to an arbitrary geometrical parameter.

A.3 Derivatives of $R_{MS}(W; x, y)$ with respect to geometrical parameters

We will show that $R_{MS}(W; x, y)$ is differentiable with respect to the geometrical parameters B, which verifies Theorem 2 in Section 4. Since the discontinuity of Haar function, the conclusion of Theorem 2 is not obvious. To achieve this goal, we will use the theory of generalized functions and generalized derivatives [2]. Following deductions are all in the sense of generalized function and generalized derivative.

We first express formally such derivatives as

$$\frac{\partial R_{MS}(W; x, y)}{\partial x_{j,i}} = \sum_{j=1}^{N} \left\{ \begin{array}{c} \sum_{k \in K} \frac{\partial}{\partial x_{j,i}} c_{0,k}^{(0,0)}(B; j) \psi_{0,k}^{(0,0)}(x, y) \\ + \sum_{s=0}^{d} \sum_{k \in K} \left[\begin{array}{c} \frac{\partial}{\partial x_{j,i}} c_{s,k}^{(0,1)}(B; j) \psi_{s,k}^{(0,1)}(x, y) \\ + \frac{\partial}{\partial x_{j,i}} c_{s,k}^{(1,0)}(B; j) \psi_{s,k}^{(1,0)}(x, y) \\ + \frac{\partial}{\partial x_{j,i}} c_{s,k}^{(1,1)}(B; j) \psi_{s,k}^{(1,1)}(x, y) \end{array} \right] \end{array} \right\}, \quad B \in \mathbb{R}^{6N}, \tag{A13}$$

and

$$\frac{\partial R_{MS}(W; x, y)}{\partial y_{j,i}} = \sum_{j=1}^{N} \left\{ \begin{array}{c} \sum_{k \in K} \frac{\partial}{\partial y_{j,i}} c_{0,k}^{(0,0)}(B; j) \psi_{0,k}^{(0,0)}(x, y) \\ + \sum_{s=0}^{d} \sum_{k \in K} \left[\begin{array}{c} \frac{\partial}{\partial y_{j,i}} c_{s,k}^{(0,1)}(B; j) \psi_{s,k}^{(0,1)}(x, y) \\ + \frac{\partial}{\partial y_{j,i}} c_{s,k}^{(1,0)}(B; j) \psi_{s,k}^{(1,0)}(x, y) \\ + \frac{\partial}{\partial y_{j,i}} c_{s,k}^{(1,1)}(B; j) \psi_{s,k}^{(1,1)}(x, y) \end{array} \right] \end{array} \right\}, \quad B \in \mathbb{R}^{6N} \tag{A14}$$

for all $j = 1, 2, \ldots, N$, $i = 1, 2, 3, 4$, and $(x, y) \in \Lambda$.

Then the remaining problem is to discuss the differentiability of Haar basis coefficients with respect to geometrical parameters, i.e., the existence of $\frac{\partial c_{s,k}^{(\cdot)}(B;j)}{\partial x_{j,i}}$ and $\frac{\partial c_{s,k}^{(\cdot)}(B;j)}{\partial y_{j,i}}$ for all $j = 1, 2, \ldots, N$, $i = 1, 2, 3, 4$, $s = 0, 1, \ldots, d$, and $k \in K$.

Generalized Derivatives of Haar Basis Functions. It is well known that the generalized derivative of $\phi(t)$:

$$\phi'(t) = \delta(t) - \delta(t - 1), \qquad t \in \mathbb{R}. \tag{A15}$$

Here δ is an impulse function satisfying:

$$\int_{-\infty}^{\infty} \delta(t) f(t) dt = f(0). \tag{A16}$$

Here $f(t)$ is an arbitrary continuous function. Note that when composed with a continuous function $g(t)$, δ holds the following property [2]:

$$\delta(g(t)) = \sum_{t_i \in T} \frac{\delta(t - t_i)}{|g'(t_i)|}, \qquad t \in \mathbb{R}. \tag{A17}$$

Here T is the set of the real roots of $g(t)$. Similarly,

$$\psi'(t) = \delta(t) - 2\delta(t - \frac{1}{2}) + \delta(t - 1), \qquad t \in \mathbb{R}. \tag{A18}$$

Therefore, for all $s \in \mathbb{Z}$, $l \in \mathbb{Z}$,

$$\begin{aligned} \phi_{s,l}'(t) &= \frac{d(\phi(2^s t - l))}{dt} \\ &= 2^s [\delta(2^s t - l) - \delta(2^s t - l - 1)] \end{aligned}, \quad t \in \mathbb{R}. \tag{A19}$$

Similarly, for all $s \in \mathbb{Z}$, $l \in \mathbb{Z}$,

$$\begin{aligned} \psi_{s,l}'(t) = 2^s [&\delta(2^s t - l) - 2\delta(2^s t - l - \frac{1}{2}) \\ &+ \delta(2^s t - l - 1))] \\ & B \in \mathbb{R}^{6N}. \end{aligned} \tag{A20}$$

Derivatives of Haar Basis Coefficients with Respect to Geometrical Parameters. We first calculate $\frac{\partial c_{s,k}^{(0,0)}}{\partial x_{j,i}}$. According to the generalized functions theory

[2], for all $j = 1, 2, \ldots, N$, $i = 1, 2, 3, 4$, $s = 0, 1, .., d$, $k_x \in K_x$ and $k_y \in K_y$,

$$\frac{\partial c_{s,k}^{(0,0)}(B;j)}{\partial x_{j,i}} = \int_0^1 \frac{\partial}{\partial x_{j,i}}[2^s \tilde{\phi}_{s,k_x}(X_j(B_x;t)) \phi_{s,k_y}(Y_j(B_y;t))Y_j'(B_y;t)]dt \quad B \in \mathbb{R}^{6N}. \tag{A21}$$

Since $\phi_{s,k_y}(Y_j(B_y;t))Y_j'(B_y;t)$ has nothing to do with the parameter $x_{j,i}$ according to Equation 2, we have

$$\begin{aligned}\frac{\partial c_{s,k}^{(0,0)}(B;j)}{\partial x_{j,i}} =& 2^s \int_0^1 \frac{\partial \tilde{\phi}_{s,k_x}(X_j(B_x;t))}{\partial x_{j,i}} \phi_{s,k_y}(Y_j(B_y;t))Y_j'(B_y;t)dt \\ =& 2^s \int_0^1 \phi_{s,k_x}(X_j(B_x;t))\frac{\partial X_j(B_x;t)}{\partial x_{j,i}} \phi_{s,k_y}(Y_j(B_y;t))Y_j'(B_y;t)dt \\ & B \in \mathbb{R}^{6N}.\end{aligned} \tag{A22}$$

Thus, the derivative of $c_{s,k}^{(0,0)}(B;j)$ with respect to any value of $x_{j,i}$ exists . Also, it can be analytically calculated by substituting Equation 2 and Equation A4 into Equation A22.

Now we turn to $\frac{\partial c_{s,k}^{(0,0)}}{\partial y_{j,i}}$. Similar to Equation A21 and A22, we have

$$\begin{aligned}\frac{\partial c_{s,k}^{(0,0)}(B;j)}{\partial y_{j,i}} =& \int_0^1 \frac{\partial}{\partial y_{j,i}}[2^s \tilde{\phi}_{s,k_x}(X_j(B_x;t)) \phi_{s,k_y}(Y_j(B_y;t))Y_j'(B_y;t)]dt \\ =& 2^s \int_0^1 \tilde{\phi}_{s,k_x}(X_j(B_x;t)) \frac{\partial}{\partial y_{j,i}}\left[\phi_{s,k_y}(Y_j(B_y;t))Y_j'(B_y;t)\right]dt \\ & B \in \mathbb{R}^{6N},\end{aligned} \tag{A23}$$

for all $j = 1, 2, \ldots, N$, $i = 1, 2, 3, 4$, $s = 0, 1, .., d$, $k_x \in K_x$ and $k_y \in K_y$. Here

$$\begin{aligned}&\frac{\partial}{\partial y_{j,i}}\left[\phi_{s,k_y}(Y_j(B_y;t))Y_j'(B_y;t)\right] \\ =& \frac{\partial \phi_{s,k_y}(Y_j(B_y;t))}{\partial y_{j,i}}Y_j'(B_y;t) + \phi_{s,k_y}(Y_j(B_y;t))\frac{\partial Y_j'(B_y;t)}{\partial y_{j,i}}\end{aligned} \tag{A24}$$

According to Equation A17 and A19 we have:

$$\begin{aligned}&\frac{\partial \phi_{s,k_y}(Y_j(B_y;t))}{\partial y_{j,i}}Y_j'(B_y;t) \\ =& \phi_{s,k}'(Y_j(B_y;t))Y_j'(B_y;t) \\ =& \begin{bmatrix} 2^s\delta(2^sY_j(B_y;t)-k_y) \\ -2^s\delta(2^sY_j(B_y;t)-k_y-1)\end{bmatrix} Y_j'(B_y;t)\frac{\partial Y_j(B_y;t)}{\partial y_{j,i}} \\ =& \sum_{t_0\in T_0}\frac{2^s\delta(t-t_0)}{|2^sY_j'(B_y;t_0)|}Y_j'(B_y;t_0)\frac{\partial Y_j(B_y;t_0)}{\partial y_{j,i}} \\ &- \sum_{t_1\in T_1}\frac{2^s\delta(t-t_1)}{|2^sY_j'(B_y;t_1)|}Y_j'(B_y;t_1)\frac{\partial Y_j(B_y;t_1)}{\partial y_{j,i}} \\ =& \sum_{t_0\in T_0}\delta(t-t_0)\text{sgn}(Y_j'(B_y;t_0))\frac{\partial Y_j(B_y;t_0)}{\partial y_{j,i}} \\ &- \sum_{t_1\in T_1}\delta(t-t_1)\text{sgn}(Y_j'(B_y;t_1))\frac{\partial Y_j(B_y;t_1)}{\partial y_{j,i}}, \\ & B \in \mathbb{R}^{6N},\end{aligned} \tag{A25}$$

for all $j = 1, 2, \ldots, N$, $i = 1, 2, 3, 4$, $s = 0, 1, .., d$, and $k_y \in K_y$. Here T_0 and T_1 are the sets of the real roots of

$$g_1(t) = 2^sY_j(B_y;t) - k_y, \qquad t \in [0,1] \tag{A26}$$

and

$$g_2(t) = 2^sY_j(B_y;t) - k_y - 1, \qquad t \in [0,1], \tag{A27}$$

respectively. Note that either $g_1(t) = 0$ or $g_2(t) = 0$ is a cubic equation in one variable (i.e., t). By substituting Equation A25 into Equation A23, there is

$$\begin{aligned}\frac{\partial c_{s,k}^{(0,0)}(B;j)}{\partial y_{j,i}} =& 2^s[\sum_{t_0\in T_0}\int_0^1 \delta(t-t_0)\tilde{\phi}_{s,k_x}(X_j(B_x;t)) \text{sgn}(Y_j'(B_y;t_0))\frac{\partial Y_j(B_y;t_0)}{\partial y_{j,i}}dt \\ &- \sum_{t_1\in T_1}\int_1^1 \delta(t-t_1)\tilde{\phi}_{s,k_x}(X_j(B_x;t)) \text{sgn}(Y_j'(B_y;t_1))\frac{\partial Y_j(B_y;t_1)}{\partial y_{j,i}}dt \\ &+ \int_0^1 \tilde{\phi}_{s,k_x}(X_j(B_x;t))\phi_{s,k_y}(Y_j(B_y;t)) \frac{\partial Y_j'(B_y;t)}{\partial y_{j,i}}dt] \\ & B \in \mathbb{R}^{6N},\end{aligned} \tag{A28}$$

for all $j = 1, 2, \ldots, N$, $i = 1, 2, 3, 4$, $s = 0, 1, .., d$, $k_x \in K_x$ and $k_y \in K_y$. From Equation A16 we have

$$\int_0^1 \delta(t-u)f(t)dt = f(u), \qquad u \in (0,1). \tag{A29}$$

Therefore Equation A28 could be written as

$$\frac{\partial c_{s,k}^{(0,0)}(B;j)}{\partial y_{j,i}} = 2^s \begin{bmatrix} \sum_{t_0 \in T_0} \tilde{\phi}_{s,k_x}(X_j(B_x;t_0))\text{sgn}(Y_j'(B_y;t_0))\frac{\partial Y_j(B_y;t_0)}{\partial y_{j,i}} \\ -\sum_{t_1 \in T_1} \tilde{\phi}_{s,k_x}(X_j(B_x;t_1))\text{sgn}(Y_j'(B_y;t_1))\frac{\partial Y_j(B_y;t_1)}{\partial y_{j,i}} \\ +\int_0^1 \tilde{\phi}_{s,k_x}(X_j(B_x;t))\phi_{s,k_y}(Y_j(B_y;t))\frac{\partial Y_j'(B_y;t)}{\partial y_{j,i}}dt \end{bmatrix} \quad B \in \mathbb{R}^{6N}, \tag{A30}$$

for all $j = 1, 2, \ldots, N$, $i = 1, 2, 3, 4$, $s = 0, 1, .., d$, $k_x \in K_x$ and $k_y \in K_y$. Therefore the derivative of $c_{s,k}^{(0,0)}(B;j)$ with respect to $y_{j,i}$ exists. And it can be analytically calculated by substituting Equation 2, Equation A4 and Equation A12 into Equation A30.

Similarly, for all $B \in \mathbb{R}^{6N}$, $j = 1, 2, \ldots, N$, $i = 1, 2, 3, 4$, $s = 0, 1, .., d$, $k_x \in K_x$ and $k_y \in K_y$, we can compute the remaining derivatives:

$$\frac{\partial c_{s,k}^{(0,1)}(B;j)}{\partial x_{j,i}} = 2^s \begin{bmatrix} -\sum_{t_0 \in T_0} \tilde{\psi}_{s,k_y}(Y_j(B_y;t_0))\text{sgn}(X_j'(B_x;t_0))\frac{\partial X_j(B_x;t_0)}{\partial x_{j,i}} \\ +\sum_{t_1 \in T_1} \tilde{\psi}_{s,k_y}(Y_j(B_y;t_1))\text{sgn}(X_j'(B_x;t_1))\frac{\partial X_j(B_x;t_1)}{\partial x_{j,i}} \\ -\int_0^1 \tilde{\psi}_{s,k_y}(Y_j(B_y;t))\phi_{s,k_x}(X_j(B_x;t))\frac{\partial X_j'(B_x;t)}{\partial x_{j,i}}dt \end{bmatrix} \tag{A31}$$

$$\frac{\partial c_{s,k}^{(0,1)}(B;j)}{\partial y_{j,i}} = -2^s \int_0^1 \psi_{s,k_y}(Y_j(B_y;t))\frac{\partial Y_j(B_y;t)}{\partial y_{j,i}} \phi_{s,k_x}(X_j(B_x;t))X_j'(B_x;t)dt, \tag{A32}$$

$$\frac{\partial c_{s,k}^{(1,0)}(B;j)}{\partial x_{j,i}} = 2^s \int_0^1 \psi_{s,k_x}(X_j(B_x;t))\frac{\partial X_j(B_x;t)}{\partial x_{j,i}} \phi_{s,k_y}(Y_j(B_y;t))Y_j'(B_y;t)dt, \tag{A33}$$

$$\frac{\partial c_{s,k}^{(1,0)}(B;j)}{\partial y_{j,i}} = 2^s \begin{bmatrix} -\sum_{t_0 \in T_0} \tilde{\psi}_{s,k_x}(X_j(B_x;t_0))\text{sgn}(Y_j'(B_y;t_0))\frac{\partial Y_j(B_y;t_0)}{\partial y_{j,i}} \\ +\sum_{t_1 \in T_1} \tilde{\psi}_{s,k_x}(X_j(B_x;t_1))\text{sgn}(Y_j'(B_y;t_1))\frac{\partial Y_j(B_y;t_1)}{\partial y_{j,i}} \\ -\int_0^1 \tilde{\psi}_{s,k_x}(X_j(B_x;t))\phi_{s,k_x}(Y_j(B_y;t))\frac{\partial Y_j'(B_y;t)}{\partial y_{j,i}}dt \end{bmatrix}, \tag{A34}$$

$$\frac{\partial c_{s,k}^{(1,1)}(B;j)}{\partial x_{j,i}} = -2^s \int_0^1 \psi_{s,k_x}(X_j(B_x;t))\frac{\partial X_j(B_x;t)}{\partial x_{j,i}} \psi_{s,k_y}(Y_j(B_y;t))Y_j'(B_y;t)dt, \tag{A35}$$

$$\frac{\partial c_{s,k}^{(1,1)}(B;j)}{\partial y_{j,i}} = 2^s \begin{bmatrix} \sum_{t_0 \in T_0} \tilde{\psi}_{s,k_x}(X_j(B_x;t_0))\text{sgn}(Y_j'(B_y;t_0))\frac{\partial Y_j(B_y;t_0)}{\partial y_{j,i}} \\ -2\sum_{t_1 \in T_1} \tilde{\psi}_{s,k_x}(X_j(B_x;t_1))\text{sgn}(Y_j'(B_y;t_1))\frac{\partial Y_j(B_y;t_1)}{\partial y_{j,i}} \\ +\sum_{t_2 \in T_2} \tilde{\psi}_{s,k_x}(X_j(B_x;t_2))\text{sgn}(Y_j'(B_y;t_2))\frac{\partial Y_j(B_y;t_2)}{\partial y_{j,i}} \\ +\int_0^1 \tilde{\psi}_{s,k_x}(X_j(B_x;t))\psi_{s,k_x}(Y_j(B_y;t))\frac{\partial Y_j'(B_y;t)}{\partial y_{j,i}}dt \end{bmatrix}, \tag{A36}$$

Note that all these derivatives of Haar basis coefficients with respect to each geometrical parameter exist, and can be calculated analytically. Therefore, derivatives of the rasterization function $R_{MS}(W;x,y)$ with respect to the ge ometrical parameters could be also analytically calculated by substituting Equation A22,A30-A36 into Equation A13 and Equation A14 respectively.

Since there exist analytic derivatives for $R_{MS}(W;x,y)$ with respect to each geometrical parameter, the differentiability of $R_{MS}(W;x,y)$ is proved, which verifies Theorem 2 in Section 4.

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